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$1/f$ noise and the level fluctuation law

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Abstract. An open system weakly coupled to a reservoir which consists of many quantum subsystems satisfying the Wigner level fluctuation law is considered. It is argued that for such systems $1/f$ noise appears as a generic phenomenon.

A commonly observed noise spectrum has $1/f$ behaviour over a broad frequency range [1]. In this paper we shall reproduce this spectrum using a quite general model of an open system weakly coupled to a reservoir. The main assumption is that the reservoir consists of many quantum subsystems which satisfy the Wigner level fluctuation law (WLFL) [2]. The WLFL is widely supported by experiments in nuclear [3] and atomic [4] physics and by numerical computations for quantum models exhibiting chaotic behavior [5].

We start with a general scheme including quantum or classical open system S weakly interacting with a reservoir R. The dynamics is governed by the Liouville superoperator (we put $\hbar = 1$)

$$\hat{H} = \hat{H}_S + \hat{H}_R + \hat{H}_{SR} \tag{1}$$

defined by the Hamiltonians H, H_S, H_R, H_{SR} respectively. Let $\langle A\rangle$ denote the average value of an observable A in the fixed stationary state ρ_0. We study the autocorrelation function $\langle X_t X_{t'}\rangle$ ($X_t = \exp(\mathrm{i}\hat{H}t)X$) of the observable X of S which satisfies the conditions: $\langle X\rangle = 0$, $\hat{H}_S X = 0$.

We have

$$\langle X_t X_{t'}\rangle = \langle X X_{t'-t}\rangle = f_x(t'-t) \tag{2}$$

and its spectrum is given by ($\omega = 2\pi f$)

$$S_x(\omega) = \mathrm{Re}\int_0^\infty f_x(t)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t. \tag{3}$$

This is a proper definition for the classical case where $\delta_x(\omega) = S_x(\omega) - S_x(-\omega) = 0$. For the quantum case $\delta_x(\omega)$ is of the order of $(\exp(-\omega/kT) - 1)$ but fortunately $|\omega/kT| \ll 1$ in all experiments on $1/f$ noise.

In order to calculate $S_x(\omega)$ we use a Mori treatment of the generalized Langevin equation [6] together with the weak coupling assumption. This assumption means that approximately $\rho_0 \simeq \rho_S \otimes \rho_R$ (uncorrelated state) and that the evolution of the 'random force' F_t ($F_0 = \dot{X}_0$) is driven by the free reservoir's dynamics $\exp(\mathrm{i}\hat{H}_R t)$. Putting the

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interaction Hamiltonian $H_{SR} = V \otimes \varphi$ (V, φ is an observable of S, R respectively) we obtain

$$\int_0^\infty f_x(t)\, \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}t = \langle X^2 \rangle \frac{1}{\mathrm{i}\omega + G(\omega)} \tag{4}$$

where

$$G(\omega) = (\langle (\hat{V}X)^2 \rangle / \langle X^2 \rangle) \int_0^\infty \langle \varphi \varphi_t \rangle\, \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}t. \tag{5}$$

Hence

$$S_x(\omega) = \langle X^2 \rangle \frac{\mu(\omega)}{(\omega + \nu(\omega))^2 + \mu^2(\omega)} \tag{6}$$

where $\mu(\omega) = \mathrm{Re}\, G(\omega)$, $\nu(\omega) = \mathrm{Im}\, G(\omega)$.

$S_x(\omega)$ exhibits $1/\omega$ behaviour for $\omega_{\min} < |\omega| < \omega_{\max}$ under the following conditions:

(I) linear shape of $\mu(\omega)$, i.e. $\mu(\omega) = \gamma|\omega|$, for $\omega_{\min} < |\omega| < \omega_{\max}$;

(II) 'fine tuning' of the renormalized frequency, i.e. $\nu(\omega) = \nu_0 + \omega\nu_1(\omega)$ such that $|\nu_0| \ll \omega_{\min}$, $|\nu_1(\omega)| \ll 1$ for $\omega_{\min} < |\omega| < \omega_{\max}$.

A typical model of R is a free Bose field in the equilibrium state representing photons, phonons etc. Moreover the coupling of S and R is assumed to be linear in field, local and of the gradient type. Therefore if we assume the linear dispersion relation $\omega(k) = v|k|$ for the field's quanta and for $|\omega| \ll kT$ we obtain: $\mu(\omega)$ is proportional to $|\omega|^{d-1}$ where d is the dimension of the configuration space. As a consequence such models may explain $1/f$ noise for two-dimensional systems only. Another argument against such models (at least for low-frequency fluctuations in solids) is that the wavelengths of photons, phonons etc corresponding to the frequency interval $[\omega_{\min}, \omega_{\max}]$ are much larger than the typical length of the sample.

We propose now a physically different realization of the free Bose field reservoir. We assume that the system S is coupled by means of a long range interaction to N identical quantum subsystems with discrete spectra. We choose the interaction Hamiltonian of the mean-field type

$$H_{SR} = V \otimes \varphi_N(Q) \tag{7}$$

with $\varphi_N(Q) = N^{-1/2} \sum_{k=1}^{N} Q^{(k)}$ scaled as fluctuations. The constituents of R are treated as independent, which implies $H_R = \sum_{k=1}^{N} h^{(k)}$ and $\rho_R = \otimes_N \rho_1$. We put also $\mathrm{tr}(\rho_1 Q) = 0$ and $\hat{h}\rho_1 = 0$. Introducing time dependence of $\varphi_N(Q)$ by $\varphi_N(Q, t) = \exp(\mathrm{i}\hat{H}_R t)\varphi_N(Q)$ one may prove that [7, 8]

$$\lim_{N\to\infty} \varphi_N(Q, t) = \varphi(Q_t) \tag{8}$$

where $\varphi(A)$ is a smeared Bose field defined on the 'single particle' Hilbert space $\mathscr{H}_1$ ($A \in \mathscr{H}_1$). $\mathscr{H}_1$ is a closure of the space of observables for a single bath constituent such that $\mathrm{tr}(\rho_1 A) = 0$ and equipped with the scalar product

$$\langle A|B \rangle = \mathrm{tr}(\rho_1 A^+ B). \tag{9}$$

The limit (8) means that the multitime correlation function for $\varphi_N(Q, t)$ tend to the vacuum expectation values for products of the related field operators $\varphi(Q, t)$. For example

$$\langle \varphi_N(Q)\varphi_N(Q, t) \rangle \to \langle 0|\varphi(Q)\varphi(Q_t)0 \rangle = \langle Q|Q_t \rangle. \tag{10}$$

The time evolution $t \to A_t$ is defined by $A_t = e^{iht} A\, e^{-iht}$. Introducing the spectral resolution of $h = \Sigma_n \varepsilon_n |n\rangle\langle n|$, $(\varepsilon_{n+1} \geq \varepsilon_n)$ and using (5), (8), (9), (10) we may write for $N \to \infty$

$$\mu(\omega) = \pi\lambda^2 \sum_{n,n'} |\langle n|Q|n'\rangle|^2 \langle n|\rho_1|n\rangle \delta[(\varepsilon_n - \varepsilon_{n'}) - \omega] \tag{11}$$

$$\nu(\omega) = \lambda^2 \sum_{n,n'} |\langle n|Q|n'\rangle|^2 \langle n|\rho_1|n\rangle \frac{1}{(\varepsilon_n - \varepsilon_{n'}) - \omega} \tag{12}$$

with $\lambda^2 = (\langle(\hat{V}X)^2\rangle / \langle X^2\rangle)$.

The next assumption is that the constituents of R are not strictly identical but they are rather described by the ensemble of Hamiltonians $\{h\}$ satisfying the WLFL. Therefore the nearest-neighbour level spacing distribution is given by [2, 3]

$$p(s) = (\pi s / 2\Omega^2) \exp(-\pi s^2 / 4\Omega^2) \tag{13}$$

where $s = (\varepsilon_{n+1} - \varepsilon_n)$ and Ω is the average level spacing. Generally in the formula (11) all the splittings between levels up to about kT will appear. However, for $|\omega| \ll \Omega$ only the nearest-neighbour levels are relevant. For example if the nearest-neighbour spacings are statistically independent (e.g. Poisson case) the probability $P(\varepsilon_{n+2} - \varepsilon_n < \Delta) \simeq P^2(\varepsilon_{n+1} - \varepsilon_n < \Delta)$ for $\Delta \ll \Omega$. For systems exhibiting WLFL we expect 'spectral rigidity' [5] which makes the contribution from non-nearest neighbours even less important. Hence for $|\omega| \ll \Omega$ and if the values of $|Q_{nn'}|$ for $n' = n+1$ are not strongly correlated with the energy differences $\varepsilon_{n+1} - \varepsilon_n$ we obtain the linear shape of $\mu(\omega)$

$$\mu(\omega) = \pi\lambda^2 \bar{Q}^2 p(|\omega|) = (\pi\lambda\bar{Q}/\Omega)^2 |\omega|. \tag{14}$$

Here $\bar{Q}^2 = \frac{1}{2} \Sigma_n [\langle n|\rho_1|n\rangle + \langle n+1|\rho_1|n+1\rangle] |Q_{n,n+1}|^2$.

We shall consider now the 'fine tuning' condition (II). In order to estimate $\nu(\omega)$ we use the formula which follows from (11) and (12) in the case of continuous $\mu(\omega)$

$$\nu(\omega) = \frac{1}{\pi} \mathcal{P} \int \frac{\mu(x)}{x - \omega} \mathrm{d}x. \tag{15}$$

Assuming that all relevant random energy levels $\{\varepsilon_1, \varepsilon_2, \ldots\}$ form a band of the width $\Delta E \ll kT$ we obtain a flat probability distribution $\langle n|\rho_1|n\rangle \simeq \langle n'|\rho_1|n'\rangle$ which leads to $\mu(\omega) \simeq \mu(-\omega)$ and hence to $\nu_0 \simeq 0$ and

$$\nu_1(\omega) = \frac{1}{\pi} \mathcal{P} \int_0^{(\Delta E)^2} \frac{\gamma(\xi)}{\xi - \omega^2} \mathrm{d}\xi \tag{16}$$

where $\gamma(\xi) = \mu(\xi^{1/2})/\xi^{1/2}$. Due to the weak coupling condition $\gamma(\xi) \ll 1$ and due to (14) $\gamma(\xi) = \gamma = (\pi\lambda\bar{Q}/\Omega)^2$ for $\xi \ll \Omega^2$. Then roughly for $\omega \ll \Delta E$

$$\nu_1(\omega) \simeq \frac{2}{\pi} \bar{\gamma} \ln \frac{\Delta E}{\omega} < \bar{\gamma} \ln \frac{\Delta E}{\omega_{\min}} \tag{17}$$

with $\bar{\gamma} = \gamma(\xi_0)$ for a certain value ξ_0 in the interval $[0, \Delta E^2]$. Summarizing, the $1/f$ noise appears for the observable X which is a constant of motion of the isolated system S and if S is weakly coupled to a special kind of reservoir R. Namely, R is an ensemble of localized quantum systems which satisfy the WLFL parametrized by the average nearest-neighbour spacing Ω and with the total energy band width ΔE. The following conditions must be satisfied to reproduce a $1/\omega$ power spectrum for X_t in the interval $[\omega_{\min}, \omega_{\max}]$:

$$\omega_{\max} \ll \Omega < \Delta E \ll kT \tag{18}$$

and

$$\omega_{\min} \simeq \Delta E \exp(-1/\bar{\gamma}) \ll kT \exp(-1/\bar{\gamma}) \tag{19}$$

where $\bar{\gamma} \ll 1$ describes the strength of the coupling between S and R. Putting the characteristic parameters for low-frequency fluctuations in solids [1] $\omega_{\min} \simeq 10^{-7}\,\mathrm{s}^{-1}$, $\omega_{\max} \simeq 10^{4}\,\mathrm{s}^{-1}$, $T \simeq 10^{2}\,\mathrm{K}$, we obtain

$$10^{-11}\,\mathrm{eV} \ll \Omega < \Delta E \ll 10^{-2}\,\mathrm{eV} \qquad 0 < \bar{\gamma} < 10^{-2}. \tag{20}$$

Hence one can see that the conditions (I), (II) in fact are not of the fine tuning type and could be realized for a broad range of parameters.

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